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On Asymptotic Solutions of Nonlinear and Linear Abel-Volterra Integral Equations. II

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Abstract

The nonlinear Abel-Volterra integral equations

$$\varphi^m(x) = \frac{ax^{\alpha(pm-l)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty)$$

with $\alpha > 0, p = -1, 0, 1, \cdots, m \neq 0, -1, -2, \cdots, l \in \mathbf{Z}$ (in particular if $m = 1$, linear equations) are considered. The asymptotic behavior of the solution $\varphi(x)$, as $x \to 0$, is obtained provided that $f(x)$ has the special power asymptotic behavior near zero.

1. Introduction

In [9], we have studied the asymptotic behavior of the solution $\varphi(x)$ of the nonlinear Abel-Volterra integral equation, as $x \to 0$

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{1.1}$$

with $\alpha > 0, m \neq 0, -1, -2, \cdots$, which is, in particular if $m = 1$, linear equations, provided that $a(x)$ and $f(x)$ have the asymptotics

$$a(x) \sim x^{\alpha pm} \sum_{k=-l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{1.2}$$

with $a_{-l} \neq 0$, and

$$f(x) \sim x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{1.3}$$

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with $f_{-n} \neq 0$, where $p = -1, 0, 1, \cdots, l, n \in \mathbf{Z}$, where $\mathbf{Z}$ is the set of integers. We have showed that under the certain assumptions on parameters m, p, l and n the solution $\varphi(x)$ of the equation (1.1) has the asymptotic expansion

$$\varphi(x) \sim \sum_{k=s}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0) \tag{1.4}$$

with $s \geqq -1$, where the coefficients φ_k are expressed via a_k and f_k.

The equation (1.1) belongs to the equation of Abel's type which has many applications (see the theory and applications in the books [3] and [13]). Especially equations of the form (1.1) are arisen in the nonlinear theory of wave propagation [7], [14] and in the nonlinear theory of water perlocation [11]. The problem to find the solution of the equation (1.1) in closed form or its asymptotic solution near zero or $d \leqq \infty$ is of importance. When $a(x)$ is a constant and $f(x) = 0$ the solution of (1.1) in closed form was found for $m > 1$ in [14] (see also [1]). The asymptotic behavior, as $x \to 0$ and $x \to \infty$, of the solution $\varphi(x)$ of the Abel-Volterra integral equation (1.1) with $a(x) = 1$ of the form

$$\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t) - [\varphi(t)]^m dt}{(x-t)^{1-\alpha}} + f(x) \quad (x > 0) \tag{1.5}$$

with $\alpha > 0, m > 1$ in the cases when $f(x)$ has the general power asymptotics near zero and infinity was studied in [6] and [12] for $\alpha = 1/2$ and in [2] for any $\alpha > 0$ (see also [4] in this connection) and several first terms of asymptotics of $\varphi(x)$ were obtained. It should be noted that the asymptotic behavior of solutions of more general nonlinear Volterra equations than (1.1) (with the kernel $a(x)(x-t)^{\alpha-1}/\Gamma(\alpha)$ being replaced by $k(x,t)$) was studied by many authors (see the results and bibliography in the book [5]), but most of the results gives only the first asymptotic term of solutions.

This paper is a continuation of the previous one [9]. Stating preliminary results of [9] in Section 2, we apply results from [9] to find the asymptotic behavior of the solution $\varphi(x)$ of the equation (1.1) with $a(x) = ax^{\alpha(pm-l)}$ $(a \neq 0)$, as $x \to 0$. Section 3 deals with the nonlinear equation

$$\varphi^m(x) = \frac{ax^{\alpha(pm-l)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{1.6}$$

with $\alpha > 0, p = -1, 0, 1, \cdots, m \neq 0, -1, -2, \cdots, l \in \mathbf{Z}$, provided that $f(x)$ has the asymptotic behavior (1.3). Section 4 is devoted to the linear equation

$$\varphi(x) = \frac{ax^{\alpha(p-l)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{1.7}$$

with $\alpha > 0, p = -1, 0, 1, \cdots, l \in \mathbf{Z}$, provided that $f(x)$ has the asymptotics

$$f(x) \sim x^{\alpha p} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{1.8}$$

with $n \in Z, f_{-n} \neq 0$. In Sections 5 we show then in some cases the asymptotic solutions coincide with the explicit solutions of the linear equations considered.

The results in Sections 4 and 5 are generalizations of some statements in [8]. The results in Section 3 are applied to find the asymptotic solutions of the nonlinear equations (1.6) with quasipolynomial free term $f(x)$ in the paper [10] where example are considered. It should be noted that in some cases asymptotic solutions $\varphi(x)$ of the nonlinear equations (1.6) give their exact solutions. In what follows $\mathbf{R}$ stands for the real number field.

2. Preliminaries

The following Lemmas were established in [9]

Lemma 1. *Let $p \in \mathbf{Z}, \alpha \in \mathbf{R}$ and $\{\varphi_k\}_{k=p}^{\infty}$ be a sequence of real numbers. If m is a real number such that $m \neq 1, 0, -1, -2, \cdots$ and*

$$\varphi(x) \sim \sum_{k=p}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{2.1}$$

then

$$\varphi^m(x) \sim x^{\alpha p m} \sum_{k=0}^{\infty} \Phi_{p,k} x^{\alpha k} \quad (x \to 0), \tag{2.2}$$

where the coefficients $\Phi_{p,k}$ are expressed via the coefficients φ_k:

$$
\begin{aligned}
\Phi_{p,0} &= \binom{m}{0} \varphi_p^m; \\
\Phi_{p,1} &= \binom{m}{1} \varphi_p^{m-1} \varphi_{p+1}; \\
\Phi_{p,2} &= \binom{m}{1} \varphi_p^{m-1} \varphi_{p+2} + \binom{m}{2} \varphi_p^{m-2} \varphi_{p+1}^2; \\
\Phi_{p,3} &= \binom{m}{1} \varphi_p^{m-1} \varphi_{p+3} + \binom{m}{2}\binom{2}{1} \varphi_p^{m-2} \varphi_{p+1}\varphi_{p+2} + \binom{m}{3} \varphi_p^{m-3} \varphi_{p+1}^3; \\
\Phi_{p,4} &= \binom{m}{1} \varphi_p^{m-1} \varphi_{p+4} + \binom{m}{2} \varphi_p^{m-2} \left[\varphi_{p+2}^2 + \binom{2}{1} \varphi_{p+1}\varphi_{p+3}\right]; \\
&\quad + \binom{m}{3}\binom{3}{1} \varphi_p^{m-3} \varphi_{p+1}^2 \varphi_{p+2} + \binom{m}{4} \varphi_p^{m-4} \varphi_{p+1}^4; \\
\Phi_{p,5} &= \binom{m}{1} \varphi_p^{m-1} \varphi_{p+5} + \binom{m}{2}\binom{2}{1} \varphi_p^{m-2} [\varphi_{p+1}\varphi_{p+4} + \varphi_{p+2}\varphi_{p+3}] \\
&\quad + \binom{m}{3} \varphi_p^{m-3} \left[\binom{3}{1} \varphi_{p+1}^2 \varphi_{p+3} + \binom{3}{2} \varphi_{p+1}\varphi_{p+2}^2\right] \\
&\quad + \binom{m}{4}\binom{4}{1} \varphi_p^{m-4} \varphi_{p+1}^3 \varphi_{p+2} + \binom{m}{5} \varphi_p^{m-5} \varphi_{p+1}^5, \qquad \textit{etc.}
\end{aligned} \tag{2.3}
$$

Lemma 2. *Let $p \in \mathbf{Z}, \alpha \in \mathbf{R}$ and $\{\varphi_k\}_{k=p}^{\infty}$ be a sequence of real numbers. If $m = 2, 3, \cdots$ and the asymptotic expansion (2.1) holds, then, as $x \to 0$,*

$$\varphi^m(x) \sim x^{\alpha pm} \sum_{r=0}^{\infty} \Phi_{p,k} x^{\alpha k}, \tag{2.4}$$

$$\Phi_{p,0} = \varphi_p^m, \quad \Phi_{p,k} = \sum_{i_0=0}^{m-1} \sum_{i_1, i_2, \cdots, i_j} \frac{m!}{i_0! i_1! i_2! \cdots i_j!} \varphi_p^{i_0} \varphi_{p+1}^{i_1} \varphi_{p+2}^{i_2} \cdots \varphi_{p+j}^{i_j}, \tag{2.5}$$

where the summation is taken over all nonnegative integers $i_1, i_2, \cdots, i_j$ such that

$$0 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_j \leqq k, \; i_0 + i_1 + \cdots + i_j = m, \; i_1 + 2i_2 + \cdots + j i_j = k. \tag{2.6}$$

3. Asymptotic Solutions of Nonlinear Equations

In this section we obtain the asymptotic solutions $\varphi(x)$ of the equation (1.6) provided that $f(x)$ has the asymptotic expansion (1.3). First we consider the equation (1.6) in the case $0 < \alpha < 1$. From Theorems 1, 2 and 7 in [9], we arrive at the following statements.

Theorem 1. *Let $0 < \alpha < 1, m \neq 1, 0, -1, \cdots$ and $n = 0, 1, 2, \cdots$, let $f(x)$ have the asymptotic expansion*

$$f(x) \sim x^{-\alpha m} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{3.1}$$

with $f_{-n} \neq 0$ and let the coefficients φ_k satisfy the relations

$$\frac{a\Gamma(\alpha k + 1)\varphi_k}{\Gamma(\alpha k + \alpha + 1)} + f_{k-n+1} = 0 \quad (k = -1, 0, \cdots, n-2) \tag{3.2}$$

$$\Phi_{-1,k-n+1} = \frac{a\Gamma(\alpha k + 1)\varphi_k}{\Gamma(\alpha k + \alpha + 1)} + f_{k-n+1} \quad (k = n-1, n, \cdots), \tag{3.3}$$

if $n > 0$ and the relations

$$\Phi_{-1,k+1} = \frac{a\Gamma(\alpha k + 1)\varphi_k}{\Gamma(\alpha k + \alpha + 1)} + f_{k+1} \quad (k = -1, 0, \cdots), \tag{3.4}$$

if $n = 0$, where $\Phi_{-1,k-n+1}$ are expressed via φ_k by (2.2) and (2.3) if $m \neq 1, 2, 3, \cdots$ (2.4) and (2.5) if $m = 2, 3, \cdots$. Then the integral equation

$$\varphi^m(x) = \frac{a x^{-\alpha(m+n)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{3.5}$$

is asymptotically solvable in the space of locally integrable functions on $(0,d)$ *with* $0 < d \leqq \infty$*, and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{3.6}$$

Theorem 2. *Let* $0 < \alpha < 1$ *and* $m \neq 1, 0, -1, \cdots$*, let* l, n *and* $(l-n)m$ *be integers such that* $l > n$ *and* $(l-n)m + n \geqq 0$*. Let* $f(x)$ *have the asymptotic expansion (3.1) and let the coefficients* φ_k *satisfy the relations*

$$\frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} = 0 \quad (k = l-n-1, l-n, \cdots, (l-n)m+l-2), \tag{3.7}$$

$$\Phi_{l-n-1,k-l+1-(l-n)m} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} \tag{3.8}$$

$$(k = (l-n)m+l-1, (l-n)m+l, \cdots),$$

if $(l-n)m+n > 0$ *and the relations*

$$\Phi_{l-n-1,k-l+1+n} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} \quad (k = l-n-1, l-n, \cdots), \tag{3.9}$$

if $(l-n)m+n = 0$*, where* $\Phi_{l-n-1,k-l+1-(l-n)m}$ *are expressed via* φ_k *by (2.2) and (2.3) if* $m \neq 1, 2, 3, \cdots$ *(2.4) and (2.5) if* $m = 2, 3, \cdots$*. Then the integral equation*

$$\varphi^m(x) = \frac{ax^{-\alpha(m+l)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{3.10}$$

is asymptotically solvable in the space of locally bounded functions on $(0,d)$ *with* $0 < d \leqq \infty$ *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=l-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{3.11}$$

Now we consider the equation (1.6) with $\alpha > 0$. From Theorems 3 - 5 and 10 in [9], we arrive at the following statements.

Theorem 3. *Let* $\alpha > 0$ *and* $m \neq 1, 0, -1, \cdots$*, and let* $p \geqq 0$ *and* $n \geqq p+1$ *be integers. Let* $f(x)$ *have the asymptotic expansion*

$$f(x) \sim x^{\alpha pm} \sum_{k=p-n+1}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{3.12}$$

with $f_{p-n+1} \neq 0$ *and let the coefficients* φ_k *satisfy the relations*

$$\frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-n+1} = 0 \quad (k = p, p+1, \cdots, n-2), \tag{3.13}$$

$$\Phi_{p,k-n+1} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-n+1} \quad (k = n-1, n, \cdots), \tag{3.14}$$

if $n > p+1$ *and the relations*

$$\Phi_{p,k-p} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-p} \quad (k = p, p+1, \cdots), \tag{3.15}$$

if $n = p+1$, *where* $\Phi_{p,k-p}$ *are expressed via* φ_k *by* (2.2) *and* (2.3) *if* $m \neq 1, 2, 3, \cdots$ (2.4) *and* (2.5) *if* $m = 2, 3, \cdots$. *Then the integral equation*

$$\varphi^m(x) = \frac{a x^{\alpha(pm-n)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{3.16}$$

is asymptotically solvable in the space of locally bounded functions on $(0, d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=p}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{3.17}$$

Theorem 4. *Let* $\alpha > 0, m \neq 1, 0, -1, \cdots$ *and* $p = 0, 1, 2, \cdots$, *let* l, n *and* $(l-n-p-1)m$ *be integers such that* $l-n-1 > p$ *and* $(l-n-p-1)m+n \geqq 0$. *Let* $f(x)$ *have the asymptotic expansion*

$$f(x) \sim x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{3.18}$$

with $f_{-n} \neq 0$, *and let the coefficients* φ_k *satisfy the relations*

$$\frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} = 0 \quad (k = l-n-1, l-n, \cdots, (l-n-p-1)m+l-2), \tag{3.19}$$

$$\Phi_{l-n-1,k-l+1-(l-n-p-1)m} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} \tag{3.20}$$

$$(k = (l-n-p-1)m+l-1, (l-n-p-1)m+l, \cdots),$$

if $(l-n-p-1)m+n > 0$ *and relations*

$$\Phi_{l-n-1,k-l+1+n} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} \quad (k = l-n-1, l-n, \cdots), \tag{3.21}$$

if $(l-n-p-1)m+n = 0$, *where* $\Phi_{l-n-1,k-l+1-(l-n-p-1)m}$ *are expressed via* φ_k *by* (2.2) *and* (2.3) *if* $m \neq 1, 2, 3, \cdots$ (2.4) *and* (2.5) *if* $m = 2, 3, \cdots$. *Then the integral equation*

$$\varphi^m(x) = \frac{a x^{\alpha(pm-l)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{3.22}$$

is asymptotically solvable in the space of locally bounded functions on $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form* (3.11).

Theorem 5. *Let* $\alpha > 0, m > 0$ *and* $p = -1, 0, 1, \cdots$ *and let* l *be an integer such that* $p+1-l > 0$ *and* $(p+1-l)/m$ *is an integer and* $q = p + (p+1-l)/m$. *Let* $f(x)$ *has the asymptotic expansion*

$$f(x) \sim x^{\alpha pm} \sum_{k=p+1-l}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{3.23}$$

with $f_{p+1-l} \neq 0$, *and let the coefficients* φ_k *satisfy the relations*

$$\Phi_{q,k+l-p-1} = f_k \quad (k = p+1-l, p-l, \cdots, q-l), \tag{3.24}$$

$$\Phi_{q,k+l-p-1} = \frac{a\Gamma[\alpha(k+l-1)+1]\varphi_{k+l-1}}{\Gamma[\alpha(k+l)+1]} + f_k \quad (k = q+1-l, q+2-l, \cdots), \tag{3.25}$$

where $\Phi_{q,k+1-p-l}$ *are expressed via* φ_k *by* (2.2) *and* (2.3) *if* $m \neq 1, 2, 3, \cdots$ (2.4) *and* (2.5) *if* $m = 2, 3, \cdots$. *Then the integral equation* (3.22) *is asymptotically solvable in the space of locally bounded functions on* $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=q}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{3.26}$$

Letting $p = -1$ and l be $-l$, we have:

Corollary 5.1. *Let* $\alpha > 0, m > 0$ *and* l *be a positive integer such that* l/m *is an integer and* $q = -1 + l/m$. *Let* $f(x)$ *has the asymptotic expansion*

$$f(x) \sim x^{-\alpha m} \sum_{k=l}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{3.27}$$

with $f_l \neq 0$ *and let the coefficients* φ_k *satisfy the relations*

$$\Phi_{q,k-l} = f_k \quad (k = l, l+1, \cdots, l+q), \tag{3.28}$$

$$\Phi_{q,k-l} = \frac{a\Gamma[\alpha(k-l-1)+1]\varphi_{k-l-1}}{\Gamma[\alpha(k-l)+1]} + f_k \quad (k = q+l+1, q+l+2, \cdots), \tag{3.29}$$

where $\Phi_{q,k-l}$ *are expressed via* φ_k *by* (2.2) *and* (2.3) *if* $m \neq 1, 2, 3, \cdots$ (2.4) *and* (2.5) *if* $m = 2, 3, \cdots$. *Then the integral equation* (3.22) *for* $p = -1$ *is asymptotically solvable in the space of locally bounded functions on* $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=q}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{3.30}$$

Corollary 5.2. *Let $\alpha > 0$ and $m = 2, 3, \cdots$, let $f(x)$ has the asymptotic expansion*

$$f(x) \sim \sum_{k=0}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{3.31}$$

with $f_0 \neq 0$ and let the coefficients φ_k satisfy the relations

$$\Phi_{0,0} = f_0, \quad \Phi_{0,k} = \frac{a\Gamma(\alpha k - \alpha + 1)\varphi_{k-1}}{\Gamma(\alpha k + 1)} f_k \quad (k = 1, 2, \cdots), \tag{3.32}$$

where $\Phi_{0,k}$ are expressed via φ_k by (2.4) and (2.5). Then the integral equation

$$\varphi^m(x) = \frac{a}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{3.33}$$

is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=0}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{3.34}$$

4. Asymptotic Solutions of Linear Equations

We obtain the asymptotic solutions $\varphi(x)$ of the linear equation (1.7) provided that $f(x)$ has the asymptotic expansion (1.7). As in Section 2, we first consider the case $0 < \alpha < 1$. From Theorem 11 and Corollaries 11.1 - 11.2 in [9], we obtain the following results.

Theorem 6. *Let $0 < \alpha < 1, l$ and n be the integers such that $l \geqq n$ and let $f(x)$ have the asymptotic expansion*

$$f(x) \sim x^{-\alpha} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.1}$$

with $f_{-n} \neq 0$.

a) *Let $l \geqq 0$ and $l \geqq n$ and the coefficients φ_k satisfy the relations*

$$\frac{a\Gamma(\alpha k + 1)\varphi_k}{\Gamma(\alpha k + \alpha + 1)} + f_{k-l+1} = 0 \quad (k = l-n-1, l-n, \cdots, 2l-n-2), \tag{4.2}$$

$$\varphi_{k-l} = \frac{a\Gamma(\alpha k + 1)\varphi_k}{\Gamma(\alpha k + \alpha + 1)} + f_{k-l+1} \quad (k = 2l-n-1, 2l-n, \cdots), \tag{4.3}$$

if $l > 0$ and the relation (4.3) if $l = 0$. Then the linear integral equation

$$\varphi(x) = \frac{a x^{-\alpha(l+1)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.4}$$

with $0 < \alpha < 1$ is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$ when $l = n$, and in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$ when $l > n$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=l-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{4.5}$$

b) *Let $0 > l \geqq n$ and the coefficients φ_k satisfy the relations*

$$\varphi_k = f_{k+1} \quad (k = -n-1, -n, \cdots, -n-l-2), \tag{4.6}$$

$$\varphi_k = \frac{a\Gamma(\alpha k + \alpha l + 1)\varphi_{k+l}}{\Gamma(\alpha k + \alpha l + \alpha + 1)} + f_{k+1} \quad (k = -n-l-1, -n-l, \cdots). \tag{4.7}$$

Then the linear integral equation (4.4) is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{4.8}$$

Remark 1. The relations (4.2) and (4.3) can be represented explicitly in the following form:

$$\varphi_{k+ls} = -\sum_{i=0}^{s} \left(\prod_{j=s-i}^{s} \frac{\Gamma[\alpha(k+1+jl)+1]}{\Gamma[\alpha(k+jl)+1]} \right) a^{-i-1} f_{k-l+1+(s-i)l}, \tag{4.9}$$

$$(k = l-n-1, l-n, \cdots, 2l-n-2; s = 0, 1, 2, \cdots).$$

Setting $l = 1$ in Theorem 6 a), we have:

Corollary 6.1. *Let $0 < \alpha < 1$ and $n \leqq 1$ be the integer and let $f(x)$ have the asymptotic expansion (4.1). Then the linear integral equation*

$$\varphi(x) = \frac{ax^{-2\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.10}$$

is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$ when $n = 1$, and in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$ when $n < 1$. Its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=-n}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{4.11}$$

where the coefficients φ_k are expressed via f_k by the relations

$$\varphi_k = -\sum_{i=0}^{n+k} \frac{\Gamma[\alpha(k+1)+1]}{\Gamma[\alpha(k-i)+1]} a^{1-i} f_{k-i} \quad (k = -n, -n+1, \cdots). \tag{4.12}$$

Setting $l = n = 0$ in Theorem 6 a), we have:

Corollary 6.2. *Let $0 < \alpha < 1$ and $f(x)$ have the asymptotic expansion*

$$f(x) \sim x^{-\alpha} \sum_{k=0}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.13}$$

with $f_0 \neq 0$ and

$$\Gamma(\alpha k + 1)a \neq \Gamma(\alpha k + \alpha + 1) \quad (k = -1, 0, 1, \cdots). \tag{4.14}$$

Then the integral equation

$$\varphi(x) = \frac{ax^{-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.15}$$

is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=-1}^{\infty} \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k+1} x^{\alpha k} \quad (x \to 0). \tag{4.16}$$

Corollary 6.3. *Let $0 < \alpha < 1, n = 0, -1, -2, \cdots$ and $f(x)$ has the asymptotic expansion (4.13) and there exists $j \in \{-n-1, -n, -n+1, \cdots\}$ such that*

$$\Gamma(\alpha j + 1)a = \Gamma(\alpha j + \alpha + 1). \tag{4.17}$$

If $f_j = 0$, then the integral equation (4.15) is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim cx^{\alpha j} + \sum_{k=-1, k\neq j}^{\infty} \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k+1} x^{\alpha k} \quad (x \to 0), \tag{4.18}$$

where c is an arbitrary real constant. If $f_j \neq 0$, then the equation (4.15) does not have any asymptotic solution of the form (4.5).

Letting $l = n$ and n be $-n$ in Theorem 6 b), we have:

Corollary 6.4. *Let $0 < \alpha < 1$ and $n = 1, 2, \cdots$, and let $f(x)$ have the asymptotic expansion*

$$f(x) \sim x^{-\alpha} \sum_{k=n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.19}$$

with $f_n \neq 0$. Then the linear integral equation

$$\varphi(x) = \frac{ax^{\alpha(n-1)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.20}$$

is asymptotically solvable in the space of locally bounded functions on $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{4.21}$$

where

$$\begin{aligned} &\varphi_k = f_{k+1} \quad (k = n-1, n, \cdots, 2n-2),\\ &\varphi_{k+jn} = \sum_{i=0}^{j-1} a^{j-i} \prod_{s=i}^{j-1} \frac{\Gamma[\alpha(sn+k)+1]}{\Gamma[\alpha(sn+k+1)+1]} f_{k+in+1} + f_{k+jn+1} \\ &\qquad\qquad (j = 1, 2, \cdots, k = n-1, n, \cdots, 2(n-1)). \end{aligned} \tag{4.22}$$

Setting $l = -1$ and $n = -1$ in Theorem 6 b), we have:

Corollary 6.5. *Let* $0 < \alpha < 1$. *If* $f(x)$ *has the asymptotic expansion*

$$f(x) \sim x^{-\alpha} \sum_{k=1}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.23}$$

with $f_1 \neq 0$, *then the linear integral equation*

$$\varphi(x) = \frac{a}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.24}$$

is asymptotically solvable in the space of locally bounded functions on $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim f_1 + \sum_{k=1}^{\infty} \left[\sum_{i=0}^{k-1} \frac{\Gamma(\alpha i+1) a^{k-i}}{\Gamma(\alpha k+1)} f_{i+1} + f_{k+1} \right] x^{\alpha k}. \tag{4.25}$$

Remark 2. Corollaries 6.2 and 6.3 coincide with Theorem 4.1 in [8].

Now we consider the equation (1.7) with $\alpha > 0$. From Theorem 12 and Corollaries 12.1 - 12.2 in [9], we obtain the following results.

Theorem 7. *Let* $\alpha > 0$ *and* $p = 0, 1, 2, \cdots$, *let* l *and* n *be integers such that* $n \leqq l - p - 1$ *and let* $f(x)$ *have the asymptotic expansion*

$$f(x) \sim x^{\alpha p} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.26}$$

with $f_{-n} \neq 0$.

c) *Let* $l-p-1 \geqq 0$ *and* $l-p-1 \geqq n$ *and the coefficients* φ_k *satisfy the relations*

$$\frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} = 0 \quad (k = l-n-1, l-n, \cdots, 2l-n-p-3), \tag{4.27}$$

$$\varphi_{k+p-l+1} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} \quad (k = 2l-n-p-2, 2l-n-p-1, \cdots), \tag{4.28}$$

if $l-p-1>0$ *and the relations* (4.28) *if* $l-p-1=0$. *Then the linear integral equation*

$$\varphi(x) = \frac{ax^{\alpha(p-l)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.29}$$

is asymptotically solvable in the space of locally bounded functions on $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form* (4.5).

d) *Let* $0 > l-p-1 \geqq n$ *and the coefficients* φ_k *satisfy the relations*

$$\varphi_k = f_{k-p} \quad (k = p-n, p-n+1, \cdots, 2p-n-l), \tag{4.30}$$

$$\varphi_{k+p-l+1} = \frac{a\Gamma(\alpha k+1)\varphi_k}{\Gamma(\alpha k+\alpha+1)} + f_{k-l+1} \quad (k = p-n, p-n+1, \cdots). \tag{4.31}$$

Then the linear integral equation (4.29) *is asymptotically solvable in the space of locally bounded functions on* $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=p-n}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{4.32}$$

Remark 3. The relations (4.27) and (4.28) can be represented explicitly in the following form:

$$\varphi_{k+s(l-p-1)} = -\sum_{i=0}^{s} \left(\prod_{j=s-i}^{s} \frac{\Gamma[\alpha(k+1+j[l-p-1])+1]}{\Gamma[\alpha(k+j[l-p-1])+1]} \right) a^{-i-1} f_{k-l+1+(s-i)(l-p-1)} \tag{4.33}$$

$$(k = l-n-1, l-n, \cdots, 2l-n-p-3;\ s = 0,1,2,\cdots).$$

Setting $p = l-2$ in Theorem 7 c), we have:

Corollary 7.1. *Let* $\alpha > 0, l = 2,3,\cdots$ *and* n *be a integer such that* $n \leqq 1$ *and let* $f(x)$ *has the asymptotic expansion*

$$f(x) \sim x^{\alpha(l-2)} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.34}$$

with $f_{-n} \neq 0$. *Then the linear integral equation*

$$\varphi(x) = \frac{ax^{-2\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.35}$$

is asymptotically solvable in the space of locally bounded functions on $(0,d)$ *with* $0 < d \leqq \infty$. *Its asymptotic solution* $\varphi(x)$ *has the form* (4.5), *where the coefficients* φ_k *are expressed via* f_k *by the relations*

$$\varphi_k = -\sum_{i=0}^{k+n+1-l} \frac{\Gamma[\alpha(k+1)+1]}{\Gamma[\alpha(k-i)+1]} a^{-i-1} f_{k+1-l-i} \quad (k = l-n-1, l-n, \cdots). \tag{4.36}$$

Setting $l-p-1 = n = 0$ in Theorem 7 c), we have:

Corollary 7.2. *Let* $\alpha > 0, l = 1, 2, \cdots$ *and* $f(x)$ *have the asymptotic expansion*

$$f(x) \sim x^{\alpha(l-1)} \sum_{k=0}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.37}$$

with $f_0 \neq 0$ *and*

$$\Gamma(\alpha k + 1)a \neq \Gamma(\alpha k + \alpha + 1) \quad (k = l-1, l, l+1, \cdots). \tag{4.38}$$

Then the integral equation

$$\varphi(x) = \frac{ax^{-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.39}$$

is asymptotically solvable in the space of locally bounded functions on $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=l-1}^{\infty} \left(1 - \frac{\Gamma(\alpha k+1)a}{\Gamma(\alpha k+\alpha+1)}\right)^{-1} f_{k-l+1} x^{\alpha k} \quad (x \to 0). \tag{4.40}$$

Corollary 7.3. *Let* $\alpha > 0, l = 1, 2, \cdots, n = 0, -1, -2, \cdots$ *and let* $f(x)$ *have the asymptotic expansion* (4.37) *and there exists* $j \in \{l-n-1, l-n, l-n+1, \cdots\}$ *such that*

$$\Gamma(\alpha j + 1)a = \Gamma(\alpha j + \alpha + 1). \tag{4.41}$$

If $f_{j-l+1} = 0$, *then the integral equation* (4.39) *is asymptotically solvable in the space of locally bounded functions on* $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim cx^{\alpha j} + \sum_{k=l-1, k\neq j}^{\infty} \left(1 - \frac{\Gamma(\alpha k+1)a}{\Gamma(\alpha k+\alpha+1)}\right)^{-1} f_{k+1-l} x^{\alpha k} \quad (x \to 0), \tag{4.42}$$

where c is an arbitrary real constant. If $f_{j-l+1} \neq 0$, then the equation (4.39) does not have any asymptotic solution of the form (4.3).

Setting $p = l - n - 1$ and n be $-n$ in Theorem 7 d), we have:

Corollary 7.4. *Let $\alpha > 0, p = 0, 1, 2, \cdots$ and $n = 1, 2, \cdots$, and let $f(x)$ have the asymptotic expansion*

$$f(x) \sim x^{\alpha p} \sum_{k=n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.43}$$

with $f_n \neq 0$. Then the linear integral equation

$$\varphi(x) = \frac{a x^{\alpha(n-1)}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.44}$$

is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=p+n}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{4.45}$$

where $\varphi_k = f_{k-p} \quad (k = p+n, p+n+1, \cdots, 2p+n-1)$ and

$$\varphi_{k+jn} = \sum_{i=0}^{j-1} a^{j-i} \left(\prod_{s=i}^{j-1} \frac{\Gamma[\alpha(k+sn)+1]}{\Gamma[\alpha(k+sn+1)+1]} \right) f_{k+in-p} + f_{k+jn-p}, \tag{4.46}$$

$$(j = 1, 2, \cdots, \ k = p+n, p+n+1, \cdots, p+2n-1).$$

Setting $p = l$ and $n = -1$ in Theorem 7 d), we have:

Corollary 7.5. *Let $\alpha > 0$ and $p = 0, 1, 2, \cdots$, and let $f(x)$ has the asymptotic expansion*

$$f(x) \sim x^{\alpha p} \sum_{k=1}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.47}$$

with $f_1 \neq 0$. Then the linear integral equation

$$\varphi(x) = \frac{a}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.48}$$

is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim f_1 x^{\alpha(p+1)} + \sum_{k=p+2}^{\infty} \left[\sum_{i=0}^{k-p-2} \frac{\Gamma[\alpha(p+1+i)+1] a^{k-p-1-i}}{\Gamma(\alpha k+1)} f_{i+1} + f_{k-p} \right] x^{\alpha k}. \tag{4.49}$$

5. Exact Solutions of Linear Equations

Now we show that in some cases the asymptotic solution $\varphi(x)$ of the linear equations (4.15) with (4.13) and (4.39) with (4.37) give exact solution. First we consider the equation (4.15). We suppose that $f(x)$ has the form

$$f(x) = f_{-n}x^{\alpha(-n-1)} + f_0(x^\alpha) \quad \text{for} \quad f_0(z) = \sum_{k=-n}^{\infty} f_{k+1}z^k \quad (n = 0, -1, -2, \cdots), \tag{5.1}$$

where $f_0(z)$ is an entire function.

If the conditions (4.14) are satisfied, then the asymptotic solution of (4.15) is given by (4.16). We denote by $g(x)$ the right hand side of (4.16) and write it in the form

$$g(x) = \left(1 - \frac{\Gamma(-\alpha n - \alpha + 1)a}{\Gamma(-\alpha n + 1)}\right)^{-1} f_{-n}x^{-\alpha n - \alpha} + g_0(x^\alpha), \tag{5.2}$$

where

$$g_0(z) = \sum_{k=-n}^{\infty} g_k z^k, \quad g_k = \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k+1}. \tag{5.3}$$

According to the formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + O\left(\frac{1}{z}\right)\right], \quad |\arg(z+a)| < \pi, \quad |z| \to \infty \tag{5.4}$$

(see, for example, [13, (1.66)]), we have

$$\left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} \sim 1, \quad k \to \infty. \tag{5.5}$$

Hence the function $g_0(z)$ in (5.3) is an entire function and the asymptotic solution (4.16) gives the explicit solution of the equation (4.15):

$$\varphi(x) = \sum_{k=-n-1}^{\infty} \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k+1}x^{\alpha k} \quad (x \to 0). \tag{5.6}$$

If the conditions (4.14) are not satisfied and there exists the number $j \in \{-n-1, -n, -n+1, \cdots\}$ such that (4.17) holds and $f_j = 0$, then the asymptotic solution (4.18) also gives the explicit solution of (4.15):

$$\varphi(x) = cx^{\alpha j} + \sum_{k=-n-1, k\neq j}^{\infty} \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k+1}x^{\alpha k}, \tag{5.7}$$

where c is an arbitrary real constant. Here

$$g_1(z) = \sum_{k=-n, k\neq j}^{\infty} g_k z^k, \quad g_k = \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k+1} \tag{5.8}$$

is also an entire function. Therefore from Corollaties 6.2 and 6.3 we arrive at the following statement.

Theorem 8. *Let $n = 0, -1, -2, \cdots, 0 < \alpha < 1$ and $f(x)$ be given by (5.1), where $f_0(z)$ is an entire function. If the condition (4.14) holds, then the equation (4.15) is solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$ when $n = 0$, and in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$ when $n < 0$. Its solution $\varphi(x)$ has the form (5.6) where $g_0(z)$ in (5.3) is an entire function. If there exists a number $j \in \{-n-1, -n, -n+1, \cdots\}$ such that (4.17) holds and $f_j = 0$, then the equation (4.15) is solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$ when $n = 0$, and in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$ when $n < 0$. Its solution $\varphi(x)$ has the form (5.7) where c is an arbitrary real constant and $g_1(z)$ in (5.8) is an entire function.*

Remark 4. Theorem 4.2 in [8] is the particular case of Theorem 8 for $n = 0$.

Using the same arguments as above we obtain from Corollaries 7.2 and 7.3 the statement similar to Theorem 8 for the integral equation (4.39) with (4.37).

Theorem 9. *Let $l = 1, 2, 3, \cdots, n = 0, -1, -2, \cdots$, $\alpha > 0$ and $f(x) = h(x^\alpha)$, where*

$$h(z) = \sum_{k=l-n-1}^{\infty} f_{k-l+1} z^k \tag{5.9}$$

is an entire function. If the conditions (4.38) hold, then the equation (4.39) is solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its solution $\varphi(x)$ has the form

$$\varphi(x) = \sum_{k=l-n-1}^{\infty} \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k-l+1} x^{\alpha k}, \tag{5.10}$$

where

$$h_0(z) = \sum_{k=l-n-1}^{\infty} h_k z^k, \quad h_k = \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k-l+1} \tag{5.11}$$

is an entire function. If there exists a number $j \in \{l-n-1, l-n, l-n+1, \cdots\}$ such that (4.41) holds and $f_{j-l+1} = 0$, then the equation (4.39) is solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its solution $\varphi(x)$ has the form

$$\varphi(x) = cx^{\alpha k} + \sum_{k=l-n-1, k \neq j}^{\infty} \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k+1-l} x^{\alpha k}, \tag{5.12}$$

where c is an arbitrary real constant and

$$h_1(z) = \sum_{k=l-n-1, k \neq j}^{\infty} h_k z^k, \quad h_k = \left(1 - \frac{\Gamma(\alpha k + 1)a}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} f_{k-l+1}, \tag{5.13}$$

is an entire function.

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